Zoom Class 21/04/2020

Total Differentiation Problems

Ques If $f(x,y)=0$, $\phi(y,z)=0$ show that

$$\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx}=\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}$$

Soln $\because f(x,y)=0$
then from 1st differential co-efficient we know that

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} \quad \text{——} (1)$$

and similarly since $\phi(y,z)=0$

$$\Rightarrow \frac{dy}{dz}=-\frac{\partial \phi}{\partial z}\Big/\frac{\partial \phi}{\partial y} \quad \text{——} (2)$$

Dividing (1) by (2) we get

$$\frac{dy}{dx}\cdot\frac{dz}{dy}=\frac{\cancel{-}\left(\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}\right)}{\cancel{-}\left(\frac{\partial \phi}{\partial z}\Big/\frac{\partial \phi}{\partial y}\right)}=\frac{\frac{\partial f}{\partial x}\times\frac{\partial \phi}{\partial y}}{\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}}$$

$$\Rightarrow \frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}\Big/\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}$$

$$\Rightarrow \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$$ Proved

Ques If the sides of a plane triangle ABC vary in such a way that its circum-radius remains constant, P. T. $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$

Soln Let 'R' be the circum-radius, and since it is a constant thus $2R = k$ (suppose)

Now, from properties of triangle we know

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = k$$

$$\Rightarrow a = k\sin A \; ; \; b = k\sin B \; ; \; c = k\sin C$$

$$\Rightarrow da = k\cos A \, dA \; ; \; db = k\sin B \, dB \text{ and}$$

$$dc = k\sin C \, dC$$

$$\Rightarrow \frac{da}{\cos A} = k\,dA \; ; \; \frac{db}{\cos B} = k\,dB \; ; \; \frac{dc}{\cos C} = k\,dC$$

$$\Rightarrow \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = k\,dA + k\,dB + k\,dC$$

$$\Rightarrow \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = k(dA + dB + dC)$$

$$\because A + B + C = \pi \text{ (constant)}$$

$$\Rightarrow dA + dB + dC = 0$$

$$\Rightarrow \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = k(0) = 0$$

Proved

Ques If $u = f(x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$, Prove that

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Soln $\because u = f(x, y)$

then $\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$ ——— ① — from total diffrentiation.

$\because x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

and $y = r\sin\theta \Rightarrow \frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

Also $\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta}$ ——— ②

Using above obtained values in Eqn ① & ② we get

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \quad \text{——— ③}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta \quad \text{——— ④}$$

Using eqn ③ & ④ in LHS we get

$$LHS = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

$$= \left\{\left(\frac{\partial f}{\partial x}\right)\cos\theta + \left(\frac{\partial f}{\partial y}\right)\sin\theta\right\}^2 + \frac{1}{r^2}\cdot\left\{\left(\frac{\partial f}{\partial x}\right)(r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)\right\}^2$$

$$= \left(\frac{\partial f}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\sin^2\theta + 2\sin\theta\cos\theta\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)$$
$$+ \frac{\cancel{r^2}}{\cancel{r^2}}\left\{-\left(\frac{\partial f}{\partial x}\right)\sin\theta + \frac{\partial f}{\partial y}(\cos\theta)\right\}^2$$

$$= \left(\frac{\partial f}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\sin^2\theta + \cancel{2\sin\theta\cos\theta\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)}$$
$$+ \left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\cos^2\theta \cancel{- 2\sin\theta\cos\theta\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)}$$

$$= \left(\frac{\partial f}{\partial x}\right)^2\left\{\sin^2\theta + \cos^2\theta\right\} + \left(\frac{\partial f}{\partial y}\right)^2\left\{\sin^2\theta + \cos^2\theta\right\}$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = RHS$$

Proved

Ques If 'u' be a homogeneous function of n^{th} degree in x, y, z and if $u = f(X, Y, Z)$ where X, Y, Z are differential co-eff. of 'u' w.r.t. x, y, z respectively, P.T.

$$X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} = \left(\frac{n}{n-1}\right)u$$

Soln $\because$ u is homogeneous, From Euler's theorem we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \quad \text{—— (1)}$$

Now to find the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ from total differentiation we get

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial x} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial y} \quad \text{—— (3)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial z} \quad \text{—— (4)}$$

$\left[\because u = f(X, Y, Z)\right]$

Substituting (2), (3), (4) in (1) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

$$\Rightarrow nu = x\left\{\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial x}\right\}$$

$$+ y\left\{\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial y}\right\}$$

$$+ z\left\{\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial z}\right\}$$

Taking $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial Y}$ and $\frac{\partial f}{\partial Z}$ common we get

$$\Rightarrow nu = \frac{\partial f}{\partial X}\left\{x\frac{\partial X}{\partial x} + y\frac{\partial X}{\partial y} + z\frac{\partial X}{\partial z}\right\} + \frac{\partial f}{\partial Y}\left\{x\frac{\partial Y}{\partial x} + y\frac{\partial Y}{\partial y} + z\frac{\partial Y}{\partial z}\right\} + \frac{\partial f}{\partial Z}\left\{x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} + z\frac{\partial Z}{\partial z}\right\}$$

Also since X, Y, Z are diffrential coeff. of 'u'

$$X = \frac{\partial u}{\partial x},\quad Y = \frac{\partial u}{\partial y},\quad Z = \frac{\partial u}{\partial z}$$

$$\Rightarrow nu = \frac{\partial f}{\partial X}\left\{x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right)\right\} + \frac{\partial f}{\partial Y}\left\{x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right\}$$

$$+ \frac{\partial f}{\partial Z}\left\{x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right\}$$

$$\Rightarrow nu = \frac{\partial f}{\partial X}\left\{x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y\partial x} + z\frac{\partial^2 u}{\partial z\partial x}\right\} + \frac{\partial f}{\partial Y}\left\{x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial z\partial y}\right\} + \frac{\partial f}{\partial Z}\left\{x\frac{\partial^2 u}{\partial x\partial z} + y\frac{\partial^2 u}{\partial y\partial z} + z\frac{\partial^2 u}{\partial z^2}\right\} \quad (5)$$

Diffrentiating (1) partially w.r.t. x, y, z respectively we get

$$\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right) + \left(y\frac{\partial^2 u}{\partial x\partial y}\right) + z\left(\frac{\partial^2 u}{\partial x\partial z}\right) = n\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + X + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = nX$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = nX - X = (n-1)X \quad \text{(6)}$$

$\underline{\underline{\text{Sly}}}$ $$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial z \partial y} = (n-1)Y \quad \text{(7)}$$

$$x\frac{\partial^2 u}{\partial x \partial z} + y\frac{\partial^2 u}{\partial y \partial z} + z\frac{\partial^2 u}{\partial z^2} = (n-1)Z \quad \text{(8)}$$

Using 6, 7, 8 in 5 we get

$$\Rightarrow nu = \frac{\partial f}{\partial X}\{(n-1)X\} + \frac{\partial f}{\partial Y}\{(n-1)Y\} + \frac{\partial f}{\partial Z}\{(n-1)Z\}$$

$$\Rightarrow \left\{X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z}\right\}(n-1) = nu$$

$$\Rightarrow X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} = \frac{nu}{(n-1)} = \left(\frac{n}{n-1}\right)u$$

<u>Proved</u>